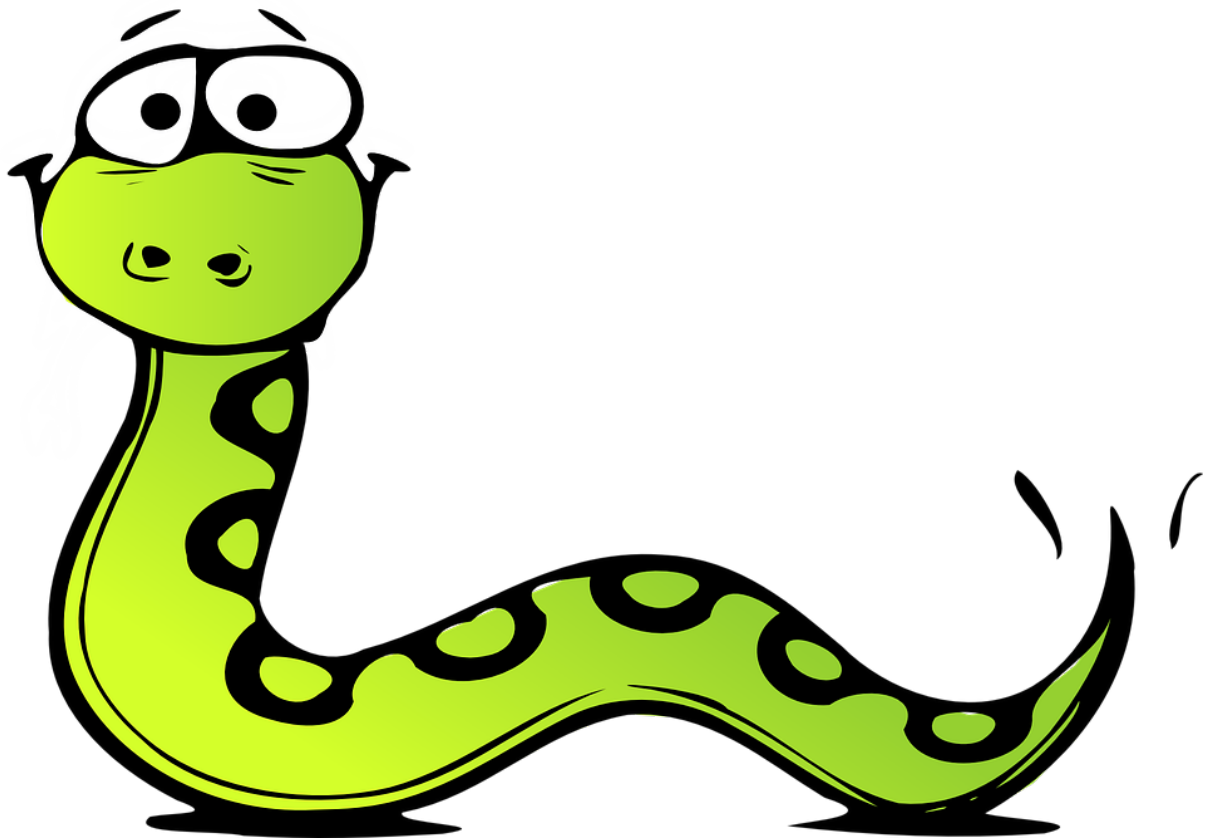


Snakes' nest



Mathematics B-day 2018



Universiteit Utrecht

Wiskunde voor
teams

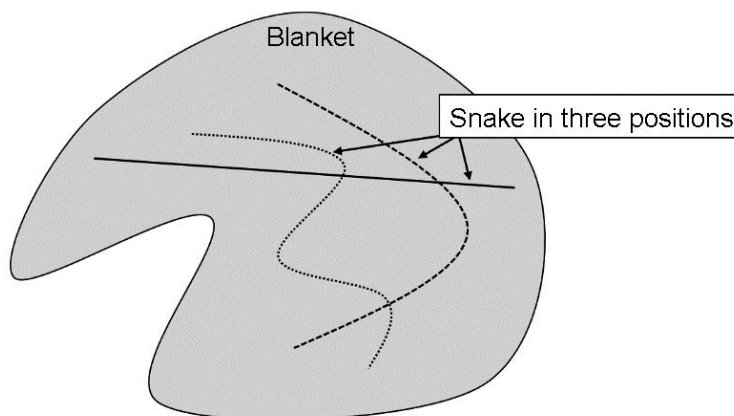


Freudenthal Institute

Introduction

About the assignment

This assignment involves a small snake, Lena, and her father, Marko. Lena is only just 15 cm long and feels so cold when she goes to sleep every night. Marko is a caring father, and he decides to knit a blanket for her. Unfortunately, the family is not well-off, and he has to economise on the yarn. That is why Marko wants to knit the smallest blanket he can. Alas, it's not possible to predict in what position Lena will fall asleep. So Marko wants to knit the blanket so it will cover Lena however she is lying. The problem you will be working on today is the following: what should the shape and area be of the smallest possible blanket that Marko can make. It is a stiff, flat blanket: you can only move it, you cannot distort or fold it.



Mathematically you can formulate the problem as follows: what is the surface of the smallest flat shape (blanket) that can cover all lines (snakes) of length 15 (after you have rotated, moved, or turned around either the snake or the blanket so that it fits). This is an open problem in mathematics: no one knows the answer!

Structure of the day

This Mathematics B-day assignment consists of introductory assignments and a final assignment. There are extra assignments: you can skip those if you want to, but they do contribute to a deeper understanding and offer extra challenges. Feel free to do a bit more work on one assignment (if you think it is an interesting problem) and a bit less on another. Try to spend around half the day on the final assignment. It consists of open research, which lets you do a lot of experimenting and reasoning. Each team can follow its own approach.

Needed materials

- Piece of copper wire, 15 cm long
- Sheets of graph paper
- Compass
- Scissors
- Adhesive tape
- Laptop/computer
- optional: GeoGebra or similar software

What do you hand in?

During the day you will work on a digital report. You will hand that in at 16:00. In it you should describe your results for all the assignments, especially the research into the final assignment. Use your own words, clearly and persuasively.

Hints:

- Plan your time and divide individual tasks up amongst the team members. It may be useful to already start writing about your work on the introductory assignments in the morning.
- *Write clearly*: so that your work can be understood by someone who did not take part in the Mathematics B-day (but who does know the necessary mathematics). This means that you must describe your work fully.
- When you provide support, clarifications or explanations, try to use *mathematical arguments* as much as possible. A combination of clarity, concision and precision is excellent!
- Use *figures* to illustrate your ideas. For example, use copies of illustrations you made (screen captures or photos of figures on paper).
- Make a *planning and divide up individual tasks* within the group.

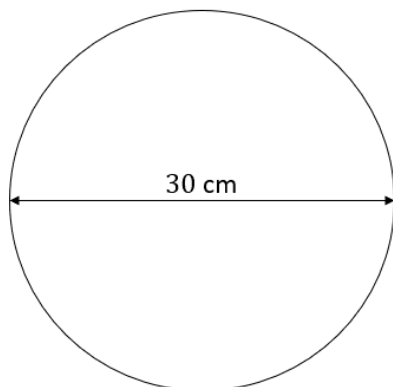
Both the mathematical content and the way the report is written will count in the mark!

Basic assignments

Assignment 1: Circular blankets

We assume that Lena is 15 cm long and super slim: she has thickness 0. She is a limber little line.

The first blanket Marko considers is circular with a 30 cm diameter.



Intuitively it is clear that Lena will always fit under it, regardless of which position she is lying in.

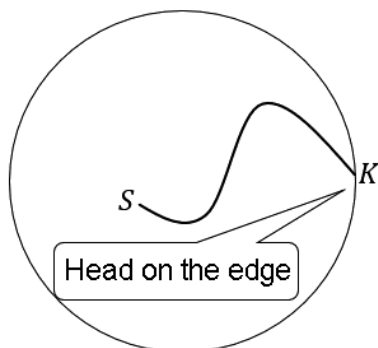
But we wonder whether a smaller diameter is also possible.

- (a) Use the copper wire you were given for this assignment to research what the smallest diameter is for a circular blanket that Lena will fit under in all positions.

Obviously, we want more from this assignment than just experimenting. We want to reason about why a blanket is or is not suitable. If you state that the snake will be covered by a specific blanket shape *in every position*, you must explain: how? How do you move the snake (without changing her shape) so that it fits?

Suppose that you have a round blanket with a 20 cm diameter. You cannot say:

"I will put Lena with her head on the edge of the circle and then rotate her around that point, until she is completely under the blanket".

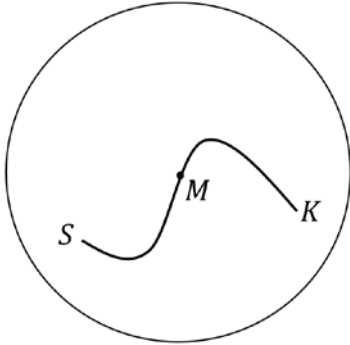


Because that will not always work!

- (b) Draw a shape for Lena where this will not work (the snake is not completely covered).

A new attempt is:

Name the centre of the circle M . Move Lena so that the middle of the snake is on M .



This will work. But now you must provide your argumentation that Lena will stay under the blanket in every position.

- (c) Do so for the circle you found in part (a).

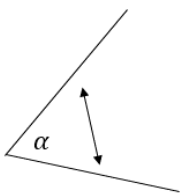
A description, that, for every position of the snake, indicates how you should position it under the blanket, is called a positioning strategy. The statements written in italic above are examples of positioning strategies.

- (d) Explain as precisely as possible why it will not fit with smaller circles than the one in part (a).
- (e) **(Extra; please note: difficult! Only do this if you have enough time).** Use the copper wire to investigate whether you can cut off another piece of the circular blanket with the smallest possible diameter (found in part (a)), so that the remaining piece is still a suitable blanket. Give a corresponding positioning strategy and explain as precisely as possible why that strategy will always work.

Assignment 2: The bending snake under a rectangular blanket

The snake can fall asleep in innumerable shapes and that is why the problem is difficult to oversee. It becomes more manageable when we look at snakes that are not as flexible as Lena.

We will therefore limit ourselves in this assignment to a rather rigid snake: consisting of two lines of length 7.5 cm with a hinge in the middle: the bending snake. The angle between the two line segments is marked with α .

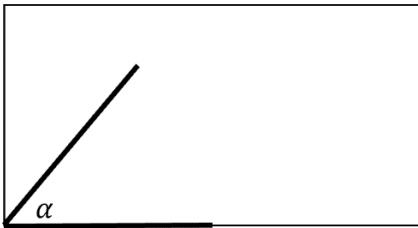


As far as blankets are concerned, at first, we will limit ourselves to rectangular blankets.

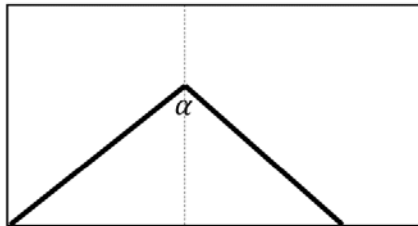
- (a) We will first look at a rectangle measuring 14 cm by 5 cm, and a rectangle measuring 15 cm by 5 cm. Use the copper wire to investigate whether the bending snake fits under these rectangular blankets in all positions. Explain your answers and, if necessary, support them with a calculation.

Marko will try to find a small rectangular blanket. He comes up with the following positioning strategy:

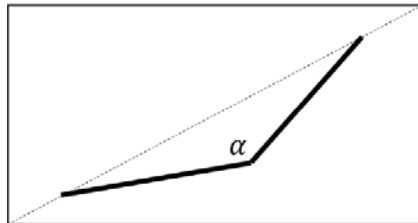
- For small angles α he places the snake with one side along the bottom edge (position 1):



- For medium angles he places the bending snake with the axis of symmetry parallel to the side of the blanket (position 2)



- For large angles he places the ends on the diagonal (position 3)



We will now elaborate this strategy with the goal of obtaining the smallest possible rectangle. It is becoming a bit more difficult now; take your time, experiment and reason about it together.

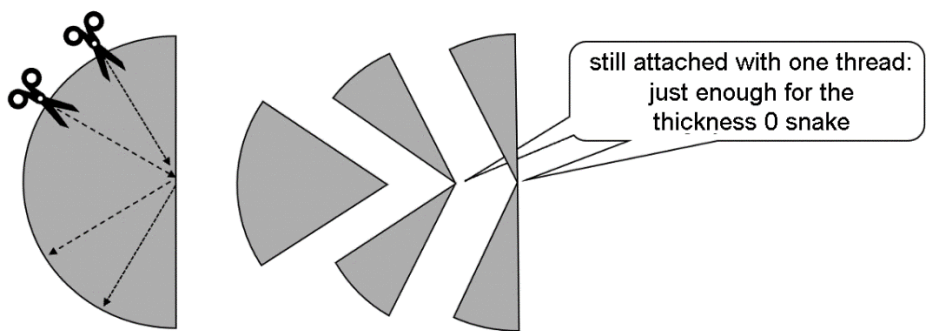
- (b) Marko initially only looks at the side (height) of the blanket. By switching from position 1 to position 2 when he has a good value for angle α , he can keep that side as short as possible. At what angle α is that? And how long will the side be??
- (c) At which angle α should Marko switch from position 2 to position 3? Explain and support with the necessary calculations!
- (d) What is the area of the smallest rectangular blanket that Marko will find like this?
- (e) Now that he has found the rectangular blanket, Marko realizes that you can cut out some parts. Do you see that as well?

Assignment 1 and 2 show a general approach to the snake-and-blanket problem. This approach returns later in the preparatory assignments and in the final assignment. We summarize this approach in the diagram below.

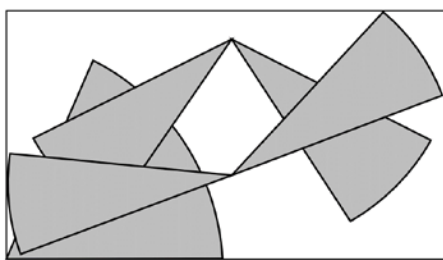
- Steps general approach**
- A. Limit yourself to a sub-problem: limit what type of snakes and what type of blankets you are considering
 - B. Experiment.
 - C. Give a *positioning strategy* for all of the snake's positions
 - D. Find the corresponding minimum dimensions
 - E. Explain that all forms remain under the blanket with the strategy from C
 - F. Optional: cut bits away from the blanket to make it even smaller

Assignment 3: Moving around fan-shaped blankets for bending snakes

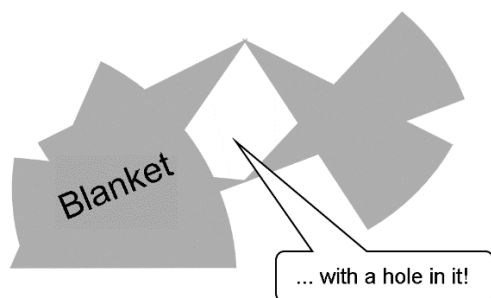
You can also try to make a blanket for bending snakes in a different way. Start with a half circle and cut it into three fan-shaped blankets, for small, medium and large angles in the bending snake.



Then try to optimally place it in a rectangle to make a smaller rectangular blanket, for example:



Even better is to cut away every part of the rectangle that does not cover anything. You get a more fanciful, but smaller blanket (maybe even with a hole in it):



The number of three fans here is arbitrary. Perhaps a *larger number* might be better.

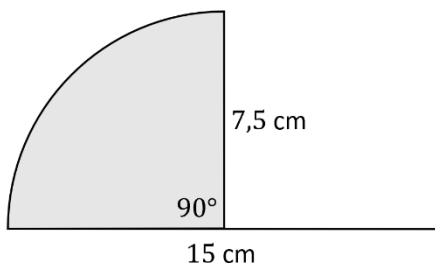
Using graph paper, make a semicircle with a radius of 7.5 cm and cut it into a number of fan blankets (use some adhesive tape to keep the corner points of two points together).

Slide the blankets over each other into a small blanket; the smaller, the better. Add a photo to your report. If you are experienced with GeoGebra or a similar program, then you can also use that. Give an explanation and, if possible, a calculation. You can also estimate the surface of the blanket, for example by counting squares.

Please note: the *type of blanket* you limit yourself to (section A of the diagram above) here is not one shape, but a number of shape that you try to shift optimally over each other! That is experimenting (part B of the scheme). Parts C, E and F are now easy, but part D can still be quite difficult.

Assignment 4: Blankets with an extra line

Another kind of blanket for **bending snakes** looks like a quarter circle (slice of pie) with a radius of 7.5 cm with a "very thin" line of 7.5 cm from the middle to the right.



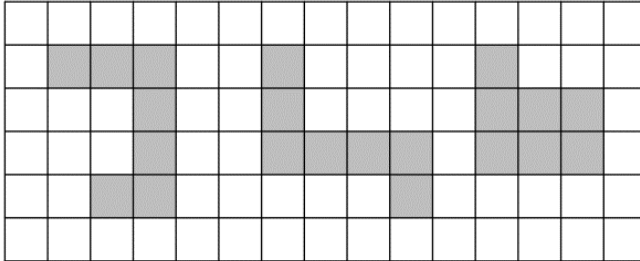
- Use the copper wire to examine how the bending snakes fit underneath. Describe your positioning strategy.
- Can you make a suitable blanket in a similar way, starting from a slice of pie with an angle of less than 90 °? And with an even smaller angle?

You will undoubtedly have discovered in part (b) that you can make the area of the blanket as small as you like. You can tell from this that we have simplified the problem with the bending snakes too much. However, it is worthwhile to first solve a difficult problem in a simplified form, because you sometimes discover techniques that also work for the non-simplified problem.

Assignment 5: Tetra snakes

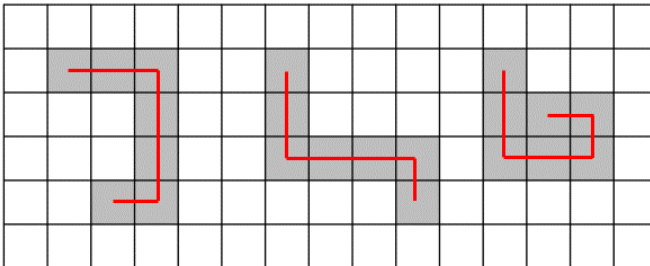
A very different way to simplify the problem is to work with snakes and blankets that consist of squares in a grid, tetra snakes and tetra blankets.

Here you can see some tetra snakes of length 7:

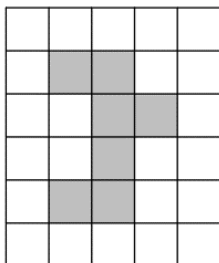


For a tetra snake every square of its body has one side touching the side of another square of its body. The squares cannot be on top of each other.

With a line through the middle of the squares you can indicate how the snake is lying; note that there are two more options for the snake on the right below.



The picture below cannot possibly be a tetra snake.

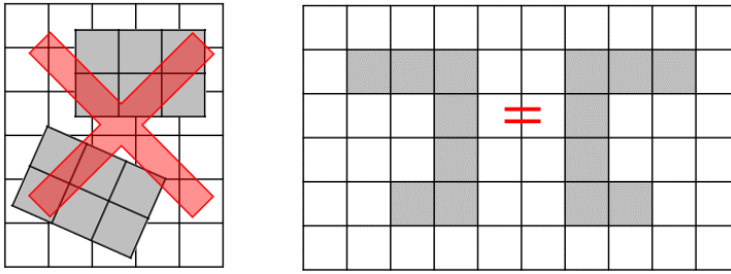


(a) Draw all possible positions that a tetra snake of length 4 can assume.

Before we can investigate what are suitable blankets for a tetra snake, we must decide on how to move the snakes (without changing their position) to fit them under the blanket. We agree on the following:

- You can slide and rotate tetra snakes, but only so that the squares of the snake fall exactly on the squares of the grid. The snake can therefore not lie *obliquely* or *half on the grid*.

- You can mirror tetra snakes, i.e. pick them up and put them on the other side.

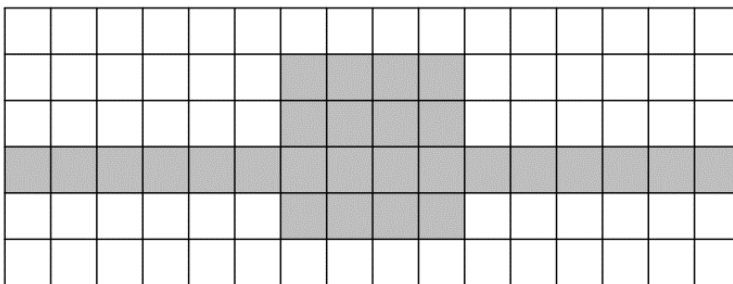


In the following sections you will design a blanket each time. Think of steps A to F of the general approach.

- (b) Find the smallest possible blanket for a tetra snake of length 4. Give the corresponding positioning strategy. Explain why, in your view, you cannot go smaller.
- (c) Find a blanket that is as small as possible for a tetra snake of length 5. Give the corresponding positioning strategy. Explain why, according to you, you cannot go smaller.
- (d) **(Extra, only if you have enough time)** Find the smallest possible blanket for a tetra snake of length 6. Give the corresponding positioning strategy. Explain why, in your view, you cannot go smaller.

Perhaps it became difficult to explain in part (d) that the blanket can really not be smaller. If your positioning strategy is good and you are sure that the snake fits under the blanket in all positions, then you know for sure that the smallest blanket is at least as small as yours. In other words, you have found an upper limit for the size of the blanket.

You can also search for lower limits. For example, you can argue that a blanket for a tetra snake of length 16 should consist of at least 28 squares: because the stretched snake and the square helical snake must fit underneath and the overlap of these two is four squares maximum, we find a minimum of $16 + 16 - 4 = 28$ squares for the blanket.



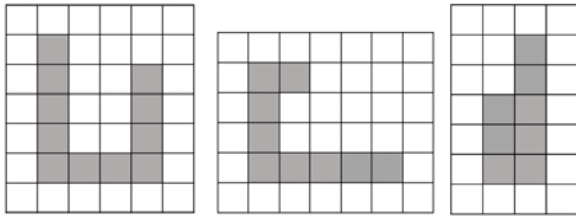
- (e) Find a higher lower limit for tetra snakes of length sixteen by adding (at least) one position of the snake to your reasoning. Then explain exactly how you got to that lower limit.

A bending tetra snake is a tetra snake with only one bend (as in Assignment 2), but now the bend is **not** necessarily in the middle.

- (f) Describe a blanket for a bending tetra snake of length n for each positive integer $n = 1, 2, 3, \dots$. Give a formula in n for the area of that blanket. Hint: first try to find the smallest blanket for small values of n (like 1 to 7) and try to make what you find there more general.

(Extra)

A U-tetra snake is a tetra snake with two 90 degree bends, so that the snake has a U-like shape. See the examples below:



The middle part (between the bends) may also have length 2, as in the example on the right.

An upper limit for the area of a smallest blanket for a tetra snake with length n is n^2 . Such a snake will fit under a square blanket with side n in any position. You can make smaller blankets especially for U-tetra snakes and so get a better upper limit!

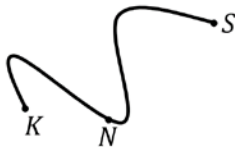
- (g) **(Extra)** Give a formula n for a better upper limit for the surface of a blanket for a U-tetra snake of length n for every positive integer $n = 1, 2, 3, \dots$

Assignment 6: a blanket for all positions

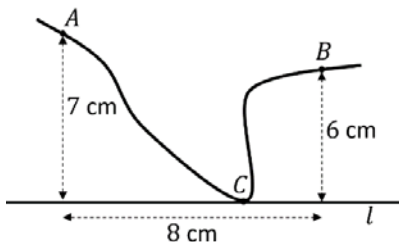
Back to the **general** problem! How can you reason out that a snake always fits under a blanket? In this assignment we make suggestions about important reasoning steps through two questions and then apply them to a blanket in the shape of a diamond.

The snake is still 15 cm long.

- (a) Suppose the head of the snake lies in point K , the point of the tail in point S and the body also passes through point N somewhere. The sum $|KN| + |NS| \leq 15$. Why?

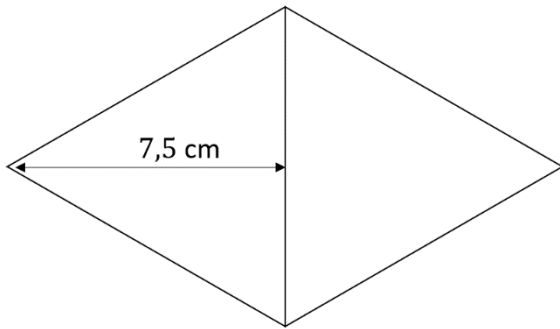


- (b) Suppose the snake passes through point A and through point B under the blanket. Line l is an edge of the blanket. The distances are as shown in the picture. Can the snake really lie so that it is on the edge of the blanket?



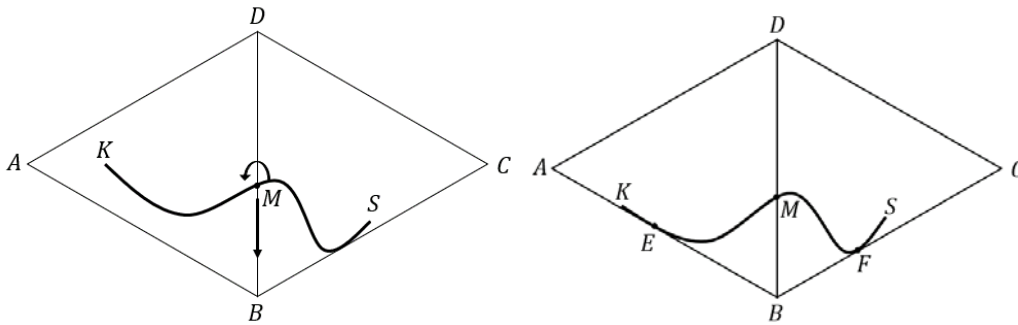
Hint: reflect point B and the piece of snake from edge point C to point B in line l .

We will keep the snake general, but we limit the shape of the blanket: a blanket consisting of two equilateral triangles with a *height* of 7.5 cm on top of each other.



(c) Calculate the area of the blanket.

Positioning strategy (see the figures below): place the snake with its middle M on the diagonal BD so that the tail part MS of the snake touches line BC (left figure). Then slide M down and rotate around M (so that tail section MS continues to touch line section BC) until the head part KM touches line segment AB (right figure). In the most extreme case, this only happens when M falls on B .



(d) Show that the snake always fits.

(e) **(Extra)** Can you cut off any pieces from the blanket?

Final assignment

The final assignment, to which you should dedicate the second part of the day, is the following: design the smallest possible blanket for the 15 cm snake. Use the general approach discussed above:

A **general approach** of the problem is the following

- A. Limit what *type* of snakes and what *type* of blankets you are considering
- B. Experiment
- C. Give a *positioning strategy* for all shapes for those blankets
- D. Find the corresponding minimum dimensions
- E. Explain that all forms remain under the blanket with the strategy from C
- F. cut bits away from the blanket to make it even smaller

Describe these steps in your report. You do not *have to* limit yourself in the way described in A. It is important that you have a clear positioning strategy and that you clearly explain that it works. On the other hand, if the calculation for point D does not work exactly, you can of course also give a good estimate; and if your reasoning for point E is not entirely successful, you can indicate which intuitions you are using, what your *hypothesis* is. You may use software such as GeoGebra.